Appendix to "Optimal Monetary Policy with $r^* < 0$"

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Abstract

We provide microfoundations to the New Keynesian model with a negative steady state real interest rate used in Billi, Galí and Nakov (2022). The model described below is a "bubbleless" version of the overlapping generations model developed in Galí (2021), augmented with discount factor shocks. It is shown to have log-linearized equilibrium conditions that take the same form as those of a standard New Keynesian model with an infinitely-lived representative consumer, but with a potentially negative steady state real interest rate.

In this Appendix to our paper "Optimal Monetary Policy with $r^* < 0$" we provide microfoundations to the log-linearized equilibrium conditions used in Billi, Galí and Nakov (2022), as well as our assumption of a negative steady state real interest rate. In particular, we show they correspond to those of a "bubbleless" version of the New Keynesian model with overlapping generations developed in Galí (2021), augmented with discount factor shocks. Our description draws heavily from that paper.

Sections 1 through 4 analyze the problems of consumers and firms, and derive the economy's equilibrum conditions. Section 5 characterizes the economy's steady state. Section 6 derives the log-linearized equilibrium conditions and shows their equivalence to those of a standard New Keynesian model with a representative household, but with a steady state real interest rate that is potentially negative, as assumed in Billi, Galí and Nakov (2022). Section 7 derives a second-order approximation to the objective function of a central bank that seeks to maximize the discounted sum of period average utilities. That approximation is shown to have a representation as a discounted sum of a linear combination of the squares of the output gap and inflation, as in the standard New Keynesian model with a representative consumer.

1 Consumers

We assume an economy with overlapping generations of the "perpetual youth" type, as in Yaari (1965) and Blanchard (1985). The size of the population is constant and normalized to one. Each individual has a constant probability γ of surviving into the following period, independently of his age and economic status ("active" or "retired"). A cohort of size $1-\gamma$ is born (in an economic sense) and becomes active each period. Thus, the size in period $t \geq s$ of the cohort born in period s is given by $(1-\gamma)\gamma^{t-s}$.

At any point in time, two types of individuals coexist in the economy, "active" and "inactive." Active individuals supply labor and manage their own firms, which they set up when they are born. We assume that each active individual faces a constant probability $1-\upsilon$ of becoming "inactive," i.e. of permanently losing his job and quitting his entrepreneurial activities. For concreteness, below we refer to the status after that transition as "retirement," though it should be clear that it can be given a broad interpretation related to skill obsolescence (due to age, health, technological or other exogenous factors). The previous assumptions imply that the size of the active population (and, hence, the measure of firms) at any point in time is constant and given by $\alpha \equiv (1-\gamma)/(1-\upsilon\gamma) \in (0,1]$.

A representative consumer from cohort s chooses a consumption plan to maximize expected lifetime utility

$$\mathbb{E}_s \sum_{t=s}^{\infty} (\beta\gamma)^{t-s} U(C_{t|s}, N_{t|s}; Z_t)$$

where $\beta \equiv \exp\{-\rho\} \in (0,1)$ is the discount factor, $C_{t|s} \equiv \left(\alpha^{-\frac{1}{\epsilon}} \int_0^\alpha C_{t|s}(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$ is a consumption index, $C_{t|s}(i)$ is the quantity consumed of good $i \in [0, \alpha]$. $N_{t|s}$ denotes work hours. Z_t is an exogenous preference shifter. Period utility is given by

$$U(C_{t|s}, N_{t|s}; Z_t) = \left(\log C_{t|s} - \frac{1}{1+\varphi} N_{t|s}\right) Z_t$$

with $z_t \equiv \log Z_t$ assumed to follow an $AR(1)$ process with zero mean and autoregressive coefficient ρ_z.

Utility maximization is subject to the sequence of period budget constraints

$$\frac{1}{P_t} \int_0^\alpha P_t(i) C_{t|s}(i) di + \mathbb{E}_t\{\Lambda_{t,t+1} \widetilde{A}_{t+1|s}\} = A_{t|s} + W_t N_{t|s} + T_t \quad (1)$$

for $t = s, s+1, s+2, ..$, where $P_t(i)$ is the price of good $i \in [0, \alpha]$, $P_t \equiv \left(\alpha^{-1} \int_0^\alpha P_t(i)^{1-\epsilon} di\right)^{\frac{1}{1-\epsilon}}$ is the price index, and W_t is the real wage. Complete markets for state-contingent securities are assumed, with $\mathbb{E}_t\{\Lambda_{t,t+1} \widetilde{A}_{t+1|s}\}$ being the market value of a portfolio of securities purchased in period t and yielding a stochastic payoff $\widetilde{A}_{t+1|s}$ at $t+1$ (expressed in units of the consumption index), where $\Lambda_{t,t+1}$ is the stochastic discount factor for one-period-ahead (real) payoffs. Variable $A_{t|s}$ denotes financial wealth at the start of period t. T_t denotes lump-sum transfers.

Only individuals who are alive can trade in securities markets. Note that the existence of complete securities markets allows individuals to insure against the loss of income due to retirement. For individuals other than those born in period t, $A_{t|s} = \widetilde{A}_{t|s}/\gamma$, where the term $1/\gamma$ captures the additional return on wealth resulting from an annuity contract. As in Blanchard (1985), that contract has the holder receive each period from a (perfectly competitive) insurance firm an annuity payment proportional to his financial wealth, in exchange for transferring that wealth to the insurance firm upon death.[1]

Both the wage and work hours are taken as given by each individual. Each firm determines the work hours it wants to hire, given desired output and technology. Aggregate work hours, N_t, are allocated uniformly among all active individuals, i.e. $N^a_{t|s} = N_t/\alpha$, with superscript a referring to an active individual. On the other hand, $N^r_{t|s} = 0$, with superscript r referring to a retired individual.

Finally, we assume a solvency constraint of the form $\lim_{T\to\infty} \gamma^T \mathbb{E}_t\{\Lambda_{t,t+T} A_{t+T|s}\} \geq 0$ for all t, where $\Lambda_{t,t+T}$ is determined recursively by $\Lambda_{t,t+T} = \Lambda_{t,t+T-1}\Lambda_{t+T-1,t+T}$.[2]

[1] Thus, individuals who hold negative assets will pay an annuity fee to the insurance company. The latter absorbs the debt in case of death. This insurance arrangement can also be replicated through securities markets.

[2] Note that $(\Lambda\gamma)^{-1}$ is the "effective" (i.e. including the impact of the annuity) interest rate paid by a borrower in the steady state. The solvency constraint thus has the usual interpretation of a no-Ponzi game condition.

The optimal allocation of expenditures yields a set of demand functions

$$C_{t|s}(i) = \frac{1}{\alpha}\left(\frac{P_t(i)}{P_t}\right)^{-\epsilon} C_{t|s} \tag{2}$$

for all $i \in [0, \alpha]$, which in turn imply $\int_0^\alpha P_{t|s}(i)C_{t|s}(i)di = P_t C_{t|s}$. The previous result, together with the assumptions made above allows to rewrite the period budget constraint as:

$$C_{t|s} + \gamma \mathbb{E}_t\{\Lambda_{t,t+1}A_{t+1|s}\} = A_{t|s} + W_t N_t \tag{3}$$

The consumer's optimal plan must satisfy the optimality condition[3]

$$\Lambda_{t,t+1} = \beta \frac{C_{t|s}}{C_{t+1|s}} \frac{Z_{t+1}}{Z_t} \tag{4}$$

and the transversality condition

$$\lim_{T\to\infty} \gamma^T \mathbb{E}_t\left\{\Lambda_{t,t+T}A_{t+T|s}\right\} = 0 \tag{5}$$

with (4) holding for all possible states of nature (conditional on the individual remaining alive in $t+1$).

1.1 Derivation of Individual Consumption Functions

The intertemporal budget constraint as of period t for an active individual born in period $s \leq t$ can be derived by iterating (3) forward from t onwards to yield:

$$\sum_{k=0}^{\infty} \gamma^k \mathbb{E}_t\{\Lambda_{t,t+k}C_{t+k|s}\} = A^a_{t|s} + \frac{1}{\alpha}\sum_{k=0}^{\infty}(\gamma\upsilon)^k \mathbb{E}_t\{\Lambda_{t,t+k}W_{t+k}N_{t+k}\} \tag{6}$$

For retired individuals the corresponding constraint is:

$$\sum_{k=0}^{\infty} \gamma^k \mathbb{E}_t\{\Lambda_{t,t+k}C_{t+k|s}\} = A^r_{t|s} \tag{7}$$

Combining (4) with (6) and (7), we obtain the individual consumption functions

$$C^a_{t|s} = (1-\beta\gamma)\widetilde{Z}_t\left[A^a_{t|s} + \frac{1}{\alpha}\sum_{k=0}^{\infty}(\upsilon\gamma)^k \mathbb{E}_t\{\Lambda_{t,t+k}W_{t+k}N_{t+k}\}\right] \tag{8}$$

$$C^r_{t|s} = (1-\beta\gamma)\widetilde{Z}_t A^r_{t|s} \tag{9}$$

for $t \geq s$, and where $\widetilde{Z}_t \equiv \frac{Z_t}{(1-\beta\gamma)\sum_{k=0}^{\infty}(\beta\gamma)^k \mathbb{E}_t\{Z_{t+k}\}}$.

[3] Note that in the optimality condition the survival probability γ and the extra return $1/\gamma$ resulting from the annuity contract cancel each other.

2 Firms

Each individual is endowed with the know-how to produce a differentiated good, and sets up a firm with that purpose at birth. That firm remains operative until its founder retires or dies, whatever comes first.[4] All firms have an identical technology, represented by the linear production function

$$Y_t(i) = N_t(i) \tag{10}$$

where $Y_t(i)$ and $N_t(i)$ denote output and employment for firm $i \in [0, \alpha]$, respectively. Individuals cannot work at their own firms, and must hire instead labor services provided by others.[5]

Aggregation of (2) across consumers yields the demand schedule facing any given firm

$$C_t(i) = \frac{1}{\alpha} \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t \tag{11}$$

where $C_t \equiv (1-\gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} C_{t|s}$ denotes aggregate consumption in period t. Each firm takes as given the aggregate price level P_t and aggregate consumption C_t.

As in Calvo (1983), each firm is assumed to freely set the price of its good with probability $1-\theta$ in any given period, independently of the time elapsed since the last price adjustment. With probability θ, an incumbent firm keeps its price unchanged, while a newly created firm sets a price equal to the economy's average price in the previous period.[6] Accordingly, the aggregate price dynamics are described by

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}$$

where P_t^* is the price set in period t by firms optimizing their price.[7] Log-linearizing the previous difference equation around the zero inflation equilibrium yields (letting lower case letters denote the logs of the original variables):

$$p_t = \theta p_{t-1} + (1-\theta) p_t^* \tag{12}$$

i.e. the current price level is a weighted average of last period's price level and the newly set price, all in logs, with the weights given by the fraction of firms that do not and do adjust prices, respectively.

In both environments, a firm adjusting its price in period t will choose the price P_t^* that maximizes

$$\max_{P_t^*} \sum_{k=0}^{\infty} (\upsilon\gamma\theta)^k \mathbb{E}_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{P_t^*}{P_{t+k}} - (1-\tau) W_{t+k} \right) \right\}$$

[4] By equating the probability of a firm's survival to that of its owner remaining active we effectively equate the rate at which dividends and labor income are discounted, which simplifies considerably the analysis below. All the qualitative results discussed below carry over to the case of different rates of "retirement" for firms and individuals, but at the cost of more cumbersome algebra.

[5] We assume that each firm newly set up in any given period inherits the index of an exiting firm.

[6] Alternatively, a fraction θ of newly created firms "inherit" the price in the previous period for the good they replace. In either case we assume a transfer system which equalizes the wealth across members of the new cohort.

[7] Note that the price is common to all those firms, since they face an identical problem.

subject to the sequence of demand constraints

$$Y_{t+k|t} = \frac{1}{\alpha}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} \tag{13}$$

for $k = 0, 1, 2, ...$where $Y_{t+k|t}$ denotes output in period $t+k$ for a firm that last reset its price in period t and τ is a constant employment subsidy.[8] Note that the $(\upsilon\gamma)^k$ component of the factor used in discounting future profits corresponds to the probability that the firm remains operative k periods ahead, while the θ^k component is the probability that the newly set price remains effective k periods ahead. Aside from the additional discounting tied to firms' finite lives, the above optimal price-setting problem is identical to that in the standard New Keynesian model, so the reader is referred to Galí (2015) for a discussion and derivation details.

The optimality condition associated with the problem above takes the form

$$\sum_{k=0}^{\infty}(\upsilon\gamma\theta)^k \mathbb{E}_t\left\{\Lambda_{t,t+k} Y_{t+k|t}\left(\frac{P_t^*}{P_{t+k}} - \mathcal{M}(1-\tau)W_{t+k}\right)\right\} = 0 \tag{14}$$

where $\mathcal{M} \equiv \frac{\epsilon}{\epsilon-1}$ is the optimal markup under flexible prices.

A first-order Taylor expansion of (14) around the zero inflation balanced growth path yields (after some algebraic manipulation):

$$p_t^* = \mu + (1-\Lambda\upsilon\gamma\theta)\sum_{k=0}^{\infty}(\Lambda\upsilon\gamma\theta)^k \mathbb{E}_t\{\psi_{t+k}\} \tag{15}$$

where $\psi_t \equiv \log((1-\tau)P_t W_t)$ is the (log) nominal marginal cost, $\mu \equiv \log \mathcal{M}$, and Λ is the value of the stochastic discount factor $\Lambda_{t,t+1}$ evaluated at the steady state. Throughout we maintain the assumption that $\Lambda\upsilon\gamma\theta \in [0, 1)$, which guarantees that the firm's problem is well defined in a neighborhood of the zero inflation steady state.[9]

Letting $\mu_t \equiv p_t - \psi_t = -\log[(1-\tau)W_t]$ denote the average (log) price markup, and combining (12) and (15) yields the inflation equation:

$$\pi_t = \Lambda\upsilon\gamma\mathbb{E}_t\{\pi_{t+1}\} - \lambda(\mu_t - \mu) \tag{16}$$

where $\pi_t \equiv p_t - p_{t-1}$ denotes inflation and $\lambda \equiv (1-\theta)(1-\Lambda\upsilon\gamma\theta)/\theta > 0$.[10]

Next, we turn to wage setting. As noted above, work hours are demand determined and allocated uniformly among active individuals. For convenience, we assume an ad-hoc wage schedule linking the

[8] The firm's demand schedule (13) can be derived by aggregating (11) across cohorts.

[9] Below we show that $\Lambda\upsilon = \beta$ must hold in the steady state, which verifies the maintained assumption.

[10] Note that in the standard NK model with a representative consumer, the coefficient on expected inflation is given by β while the slope coefficient is $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Those expressions correspond to the limit of the expressions in the text as $\upsilon\gamma \to 1$, and given that $\Lambda = \beta$ under the assumption of an infinitely-lived representative consumer.

real wage W_t to the average consumption and work hours of active individuals:

$$W_t = \Theta \frac{C_t}{\alpha} \left(\frac{N_t}{\alpha} \right)^{\varphi} \tag{17}$$

where $N_t \equiv \int_0^{\alpha} N_t(i) di$ denotes aggregate work hours and α is the aggregate labor supply. The wage is taken as given by firms.[11] Equivalently, and using the fact that $Y_t = N_t = C_t$ in equilibrium, we can rewrite (17) as:

$$W_t = \Theta \left(\frac{Y_t}{\alpha} \right)^{1+\varphi} \tag{18}$$

Wage schedule (17) and production function (10), together with the assumptions of a constant flexible price markup $\mathcal{M}$ and a constant employment subsidy τ, jointly imply a constant natural (i.e. flexible price) level of output given by $Y_t^n = \alpha((1-\tau)\mathcal{M}\Theta)^{-\frac{1}{1+\varphi}} \equiv Y^n$ for all t.

Taking logs on (17), and combining the resulting expression with $\mu_t = -\log[(1-\tau)W_t]$ and (16), we obtain a version of the New Keynesian Phillips curve

$$\pi_t = \Lambda \upsilon \gamma \mathbb{E}_t\{\pi_{t+1}\} + \kappa \widehat{y}_t \tag{19}$$

where $\kappa \equiv \lambda(1+\varphi)$, and $\widehat{y}_t \equiv \log(Y_t/Y)$ is the output gap. Note that, in contrast with the standard New Keynesian model, the coefficient on expected inflation is not pinned down by the consumer's discount factor. Instead it depends on parameters affecting the life expectancy of firms (through $\upsilon\gamma$), as well as the steady state discount factor Λ, all of which determine the effective "forward-lookingness" of price-setting.

3 Asset Markets

In addition to annuity contracts and a complete set of state-contingent securities, we assume the existence of markets for some other specific assets, whose prices and returns must satisfy certain equilibrium conditions.

In particular, let $Q_t^B \equiv \exp\{-i_t\}$ denote the price of a one-period nominally riskless pure discount bond, with i_t denoting the corresponding yield. Thus, we must have[12]

$$Q_t^B = \mathbb{E}_t \left\{ \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} \right\} \tag{20}$$

thus implying the steady state relation $\Lambda \equiv \exp\{-r^*\}$, where r^* denotes the real return on the riskless nominal bond in steady state.

[11] Note that $\frac{C_t}{\alpha}\left(\frac{N_t}{\alpha}\right)^{\varphi}$ is the *average* marginal rate of substitution between consumption and work hours across active individuals, so Θ can be interpreted as an average wage markup. Below we make assumptions on Θ that guarantee the wage is above the marginal rate of substitution *for all* individuals.

[12] Note also that in the asset pricing equations, and from the viewpoint of an individual investor, the probability of remaining alive γ and the extra return $1/\gamma$ resulting from the annuity contract cancel each other.

Stocks in individual firms trade at a price (before dividends) $Q_t^F(i)$, for all $i \in [0, \alpha]$, which must satisfy the equilibrium condition:

$$Q_t^F(i) = D_t(i) + \upsilon\gamma\mathbb{E}_t\left\{\Lambda_{t,t+1}Q_{t+1}^F(i)\right\} \tag{21}$$

where $D_t(i) \equiv Y_t(i)\left(\frac{P_t(i)}{P_t} - (1-\tau)W_t\right)$ denotes firm i's dividends, and $\upsilon\gamma$ is the probability that any firm survives into next period. Solving (21) forward under the assumption that $\lim_{k\to\infty}(\upsilon\gamma)^k\mathbb{E}_t\left\{\Lambda_{t,t+k}Q_{t+k}(i)\right\} = 0$, and aggregating across firms we obtain:

$$\begin{aligned} Q_t^F &\equiv \int_0^\alpha Q_t^F(i)di \\ &= \sum_{k=0}^{\infty}(\upsilon\gamma)^k\mathbb{E}_t\{\Lambda_{t,t+k}D_{t+k}\} \end{aligned} \tag{22}$$

where $D_t \equiv \int_0^\alpha D_t(i)di$ denotes aggregate dividends. Note that the fact that individual firms are finitely-lived makes it possible for the aggregate value of currently traded firms to be finite even if the interest rate were to be negative. Note also that, in contrast with Galí (2021), we are abstracting from the possibility of a bubble component in stock prices.

4 Market Clearing

Goods market clearing requires $Y_t(i) = (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s}(i)$ for all $i \in [0, \alpha]$. Letting $Y_t \equiv \left(\alpha^{-\frac{1}{\epsilon}}\int_0^\alpha Y_t(i)^{1-\frac{1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}$ denote aggregate output, we have:

$$\begin{aligned} Y_t &= (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s} \\ &= C_t \end{aligned}$$

Note also that in equilibrium

$$\begin{aligned} N_t &= \int_0^\alpha N_t(i)di \\ &= \Delta_t^p Y_t \end{aligned}$$

where $\Delta_t^p \equiv \frac{1}{\alpha}\int_0^\alpha (P_t(i)/P_t)^{-\epsilon}di$ is an index of relative price distortions which, up to a first-order approximation, equals unity near a zero inflation steady state.

Asset market clearing requires

$$(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}(\upsilon^{t-s}A_{t|s}^a + (1-\upsilon^{t-s})A_{t|s}^r) = Q_t^F \tag{23}$$

Aggregation of consumption functions (8) and (9) across individuals and cohorts, combined with asset market clearing condition (23), and the expression for firms' market value (22) yields the aggregate consumption function:

$$\begin{aligned} C_t &= (1-\beta\gamma)\widetilde{Z}_t \left[Q_t^F + \sum_{k=0}^{\infty} (\upsilon\gamma)^k \mathbb{E}_t\{\Lambda_{t,t+k} W_{t+k} N_{t+k}\} \right] \\ &= (1-\beta\gamma)\widetilde{Z}_t \left[\sum_{k=0}^{\infty} (\upsilon\gamma)^k \mathbb{E}_t\{\Lambda_{t,t+k} Y_{t+k}\} \right] \\ &= (1-\beta\gamma)\widetilde{Z}_t X_t \end{aligned} \tag{24}$$

where

$$X_t \equiv \sum_{k=0}^{\infty} (\upsilon\gamma)^k \mathbb{E}_t\{\Lambda_{t,t+k} Y_{t+k}\} \tag{25}$$

can be interpreted as total wealth (i.e. the discounted sum of current and expected future income) of individuals currently alive. Note that we can rewrite (25) in recursive form as:

$$X_t \equiv \upsilon\gamma \mathbb{E}_t\{\Lambda_{t,t+1} X_{t+1}\} + Y_t \tag{26}$$

Next, we characterize the economy's steady state consistent with zero inflation.

5 Steady State

In a perfect foresight steady state, the discount factor is constant and satisfies $\Lambda = \exp\{-r^*\}$, as implied by (20), where r^* denotes the steady state real interest rate. Note also that a steady state with zero inflation requires that actual and desired markups coincide, i.e. $(1-\tau)W = 1/\mathcal{M}$. Combined with the wage rule (18), the previous condition implies that steady state output Y coincides with the (constant) natural level of output $Y = Y^n = \alpha((1-\tau)\mathcal{M}\Theta)^{-\frac{1}{1+\varphi}}$, as derived above.

Evaluating (24) and (25) at the steady state, and noting that in the latter $\widetilde{Z} = 1$, yields

$$C = \frac{1-\beta\gamma}{1-\Lambda\upsilon\gamma} Y \tag{27}$$

where C denotes aggregate consumption evaluated at the steady state. Goods market clearing requires that $C = Y$ thus implying $\Lambda\upsilon = \beta$. Equivalently,

$$r^* = \rho + \log \upsilon$$

Note that the steady state real interest rate is increasing in υ. The reason is that an increase in that parameter raises desired consumption by increasing the expected stream of future income for currently active individuals, for any given level of aggregate output. In order for the goods market to clear, an increase in the interest rate is required.

When $\upsilon = 1$ (i.e., no retirement) the steady state real interest rate is pinned down by the discount rate, i.e. $r^* = \rho > 0$, as in the standard representative agent model, and is thus constrained to be positive. More generally, r^* becomes negative if and only if $\upsilon < \beta$. This is thus the case consistent with the analysis in Billi, Galí and Nakov (2022).[13]

The key role of retirement or, more generally, the anticipation of declining relative income in bringing about an interest rate lower than the growth rate was a central theme in Blanchard (1985) in a deterministic OLG model.[14]

6 Log-linearized Equilibrium Conditions around the Steady State

Given the steady state relation $\Lambda\upsilon = \beta$, we can rewrite the New Keynesian Phillips curve (19) as

$$\pi_t = \beta\gamma\mathbb{E}_t\{\pi_{t+1}\} + \kappa\widehat{y}_t \tag{28}$$

which takes the same form as in the standard NK model, and as equation (1) in Billi, Galí and Nakov (2022), with the discount factor in the latter suitably redefined.

Log-linearization of the bond-pricing equation (20) yields:

$$-\mathbb{E}_t\{\widehat{\lambda}_{t,t+1}\} = i_t - \mathbb{E}_t\{\pi_{t+1}\} - r^* \equiv \widehat{r}_t$$

Furthermore, log-linearization of the aggregate consumption function (24) yields:

$$\widehat{c}_t = \widehat{x}_t + \frac{\beta\gamma(1-\rho_z)}{1-\beta\gamma\rho_z} z_t \tag{29}$$

where $\widehat{x}_t \equiv \log(X_t/X)$ and $X(1-\beta\gamma) = Y$. On the other hand, log-linearization of (26) around the steady state yields

$$\widehat{x}_t = \beta\gamma\mathbb{E}_t\{\widehat{x}_{t+1}\} - \beta\gamma\widehat{r}_t + (1-\beta\gamma)\widehat{y}_t \tag{30}$$

[13] Note also that a change in the expected lifetime, as indexed by γ, does not have an independent effect on r. The reason is that, when $\Lambda\upsilon = \beta$, a change in γ scales in the same proportion the present value of consumption and that of income, leaving aggregate consumption unchanged and making an adjustment in the real rate unnecessary. The independence of the steady state real interest rate from γ is a consequence of the log utility specification assumed here. That property is not critical from the viewpoint of the present paper, since there are other factors (the probability of retirement, in particular), that can drive the real interest rate towards negative values.

[14] In the classical OLG framework with two-period lives, the assumption of declining labor income, usually in the form of a lower endowment or no labor supply for the old, plays a key role in lowering the real interest rate below the growth rate, thus creating the conditions for the emergence of bubbles.

Combining (29) and (30) we can write the aggregate consumption function as:

$$\widehat{c}_t = \beta\gamma\mathbb{E}_t\{\widehat{c}_{t+1}\} - \beta\gamma\widehat{r}_t + (1-\beta\gamma)\widehat{y}_t + \beta\gamma(1-\rho_z)z_t$$

Imposing the goods market clearing condition $\widehat{c}_t = \widehat{y}_t$ for all t and rearranging terms yields the dynamic IS equation:

$$\widehat{y}_t = \mathbb{E}_t\{\widehat{y}_{t+1}\} - (i_t - \mathbb{E}_t\{\pi_{t+1}\} - r_t^n)$$

with $r_t^n = r^* + (1-\rho_z)z_t$ and with $r^* < 0$ under the assumption that $\upsilon < \beta$. This representation corresponds to (2) in Billi, Galí and Nakov (2022) under the assumption that $\sigma = 1$ and after an innocuous rescaling of z_t.

7 Welfare

In the present section we provide a welfare-theoretical justification for the central bank loss function assumed in Billi, Galí and Nakov (2022). We start by deriving an expression for average utility across individuals alive in period t, denoted by U_t, as a function of aggregate variables. Before we carry out that derivation we take a brief detour to show how individual consumption relates to aggregate consumption over an individual's lifetime.

7.1 The Evolution of Relative Consumption

For the purposes of this section, and for analytical convenience, we assume a self-financing transfer scheme that equates the financial wealth of all newly born consumers, independently of whether their firm (which is their only asset when born) optimizes or not the price in its first period of operations. Under that assumption, the financial wealth of a newly born individual is given by $A_{t|t}^a = \frac{1}{\alpha}Q_t^F$ where Q_t^F is the aggregate market value of firms operating in period t. Thus, evaluating (8) for $s = t$ and imposing the previous assumption we can write:

$$\begin{aligned} C_{t|t}^a &= \left(\frac{1-\beta\gamma}{\alpha}\right)\widetilde{Z}_t\left[Q_t^F + \sum_{k=0}^{\infty}(\upsilon\gamma)^k\mathbb{E}_t\{\Lambda_{t,t+k}W_{t+k}N_{t+k}\}\right] \\ &= \left(\frac{1-\beta\gamma}{\alpha}\right)\widetilde{Z}_tX_t \end{aligned} \tag{31}$$

Combining (24) and (31) implies that consumption of the newly born must satisfy:

$$C_{t|t}^a = \frac{1}{\alpha}C_t \tag{32}$$

Furthermore, we can write

$$\begin{aligned} C_{t+1} &= (1-\gamma)C_{t+1|t+1} + (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t+1-s}C_{t+1|s} \\ &= (1-\upsilon\gamma)C_{t+1} + (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t+1-s}C_{t+1|s} \\ &= \frac{1-\gamma}{\upsilon}\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t+1|s} \\ &= \frac{1-\gamma}{\upsilon}\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s}\frac{\beta}{\Lambda_{t,t+1}}\frac{Z_{t+1}}{Z_t} \\ &= \frac{1}{\upsilon}\frac{\beta}{\Lambda_{t,t+1}}\frac{Z_{t+1}}{Z_t}C_t \end{aligned} \tag{33}$$

where the second equality makes use of (32) and the fourth equality invokes the optimality condition (4).

Combining (4) and (33) we obtain the following law of motion for the relative consumption of a household of a given cohort:

$$\frac{C_{t+1|s}}{C_{t+1}} = \upsilon\frac{C_{t|s}}{C_t}$$

from which it follows that

$$\frac{C_{t+k|s}}{C_{t+k}} = \upsilon^k\frac{C_{t|s}}{C_t}$$

for $k = 0, 1, 2, ...$ and for all $t \geq s$.

Evaluating the previous expression at $s = t$ we obtain:

$$\begin{aligned} \frac{C_{t+k|t}}{C_{t+k}} &= \upsilon^k\frac{C_{t|t}}{C_t} \\ &= \upsilon^k\frac{1}{\alpha} \end{aligned} \tag{34}$$

where the second equality follows from (32).

7.2 An Objective Function for the Central Bank

We define average period t utility across individuals alive as follows

$$\begin{aligned} U_t &= \left[(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}\log C_{t|s} - \frac{\alpha}{1+\varphi}\left(\frac{N_t}{\alpha}\right)^{1+\varphi}\right]Z_t \\ &= \left[(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}\log\left(\frac{\upsilon^{t-s}C_t}{\alpha}\right) - \frac{\alpha}{1+\varphi}\left(\frac{N_t}{\alpha}\right)^{1+\varphi}\right]Z_t \\ &= \left[\log C_t - \frac{1}{(1+\varphi)\alpha^{\varphi}}N_t^{1+\varphi}\right]Z_t + t.i.p. \end{aligned}$$

Assuming the same discount factor for the central bank as for individual consumers, and ignoring terms independent from policy, we obtain an objective function for the central bank, expressed in terms of aggregate variables, given by:

$$\mathbb{L} \equiv \sum_{t=0}^{\infty} (\beta\gamma)^t V(C_t, N_t; Z_t)$$

where $V(C, N; Z) \equiv \left[\log C - \frac{1}{(1+\varphi)\alpha^{\varphi}} N^{1+\varphi}\right] Z$.

Next, we derive a second order approximation to the previous objective function. In doing so, and following conventional practice, we assume that the employment subsidy τ is chosen to guarantee that the steady state output is given by $Y = \alpha^{\frac{\varphi}{1+\varphi}}$, which is the level that maximizes period utility $\log Y - \frac{1}{(1+\varphi)\alpha^{\varphi}} Y^{1+\varphi}$. This requires that $\alpha(1-\tau)\mathcal{M}\Theta = 1$.[15]

Thus, and up to a second order approximation, in a neighborhood of the optimal steady state, we have:

$$\begin{aligned}
V_t - V &\simeq \left(\frac{C_t - C}{C}\right)(1+z_t) - \frac{1}{2}\left(\frac{C_t - C}{C}\right)^2 - \left(\frac{N_t - N}{N}\right)(1+z_t) - \frac{\varphi}{2}\left(\frac{N_t - N}{N}\right)^2 \\
&\simeq \left(\widehat{y}_t + \frac{1}{2}\widehat{y}_t^2\right)(1+z_t) - \frac{1}{2}\widehat{y}_t^2 - \left(\widehat{n}_t + \frac{1}{2}\widehat{n}_t^2\right)(1+z_t) - \frac{\varphi}{2}\widehat{n}_t^2 \\
&\simeq \widehat{y}_t(1+z_t) - (\widehat{y}_t + \log \Delta_t^p)(1+z_t) - \frac{1+\varphi}{2}\widehat{y}_t^2 \\
&\simeq -\frac{\epsilon}{2} var_i\{p_t(i)\} - \frac{1+\varphi}{2}\widehat{y}_t^2
\end{aligned}$$

where we have used the approximation $\log \Delta_t^p \simeq \frac{\epsilon}{2} var_i\{p_t(i)\}$ as derived in Woodford (2003, chapter 6). Similarly, as proved in Woodford (2003, chapter 6), we have

$$\sum_{t=0}^{\infty} (\beta\gamma)^t var_i\{p_t(i)\} \simeq \frac{1}{\lambda} \sum_{t=0}^{\infty} (\beta\gamma)^t \pi_t^2$$

It follows that in a neighborhood of the optimal steady state

$$\begin{aligned}
\sum_{t=0}^{\infty} (\beta\gamma)^t (V - V_t) &\simeq \frac{1}{2} \sum_{t=0}^{\infty} (\beta\gamma)^t \left[(1+\varphi)\widehat{y}_t^2 + \frac{\epsilon}{\lambda}\pi_t^2\right] \\
&= \frac{\epsilon}{2\lambda} \sum_{t=0}^{\infty} (\beta\gamma)^t \left(\vartheta \widehat{y}_t^2 + \pi_t^2\right)
\end{aligned}$$

where $\vartheta \equiv \frac{\kappa}{\epsilon}$. The last expression corresponds to the loss function used in Billi, Galí and Nakov (2022), up to multiplicative scalar and after a suitable reinterpretation of the discount factor.

[15] Note that in the limiting case of an infinitely-lived representative consumer ($\alpha = 1$) and perfectly competitive labor markets ($\Theta = 1$) the previous condition takes the familiar form $(1-\tau)\mathcal{M} = 1$.

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